BASIC LOGIC AND TRUTH TABLES

MATT LEONARD (USC)

The point of this handout is to teach you how to check whether or not an argument is valid. An argument is valid just in case there is no way the premises could be true while the conclusion be false. A good argument is both valid and sound, but checking for validity can initially be a bit confusing, until you get the hang of it. If you're not sure what an argument is or what validity is, you should read the first half of my handout "Two Tips for Taking Philosophy Courses," available here: http://www.matt-leonard.org/teaching.html

In this handout, you will learn how to check for the validity or invalidity of an argument by using *the truth-table method*. Learning this method will require two important steps. First, you will need to learn a new (but very basic) language called the *language of propositional logic* (from here on out, let's call this language PL). Second, you will need to learn how to use *truth-tables* to check an argument for validity or invalidity.

1 THE LANGUAGE OF PROPOSITIONAL LOGIC

First, you need to learn a little language: PL. There are two central building blocks for sentences in PL:

1. Sentence Variables: $A, B, C, ..., Z$
2. The 5 Connectives and Their Meanings:
   a) & = ... and ...
   b) $\vee$ = ... or ...
   c) $\rightarrow$ = If ..., then ...
   d) $\leftrightarrow$ = ... if and only if ....
   e) $\sim$ = Not ...

Sentence variables represent certain simple sentences in English, sentences which don't contain words like 'and', 'if', 'or', or 'not'. Sentence variables represent simple sentences like the following: "God exists," "abortion before 20 weeks is morally permissible," "humans are morally responsible for their actions," and so forth.

Let's pretend for a moment that P, Q, and R have the following meanings:

- P = It is raining.
- Q = Los Angeles is cloudy.

- R = Taylor Swift has the deepest song lyrics.

We build up more complex sentences of PL by taking P, Q and R and combing them with the connectives. So, given the meanings we've given above to P, Q, and R, here are some complex sentences and their meanings.

- $P\&Q$ = It is raining and Los Angeles is cloudy.
- $\sim Q\&P$ = Los Angeles isn't cloudy and it is raining.
- $R \vee Q$ = Either Taylor Swift has the deepest song lyrics or Los Angeles is cloudy.
- $\sim R \leftrightarrow \sim P$ = Taylor Swift doesn't have the deepest song lyrics if and only if it's not raining.
- $\sim\sim R$ = It's not the case that Taylor swift doesn't have the deepest song lyrics (but this just means: Taylor swift has the deepest song lyrics).

Perhaps you noticed one important difference between $\&, \vee, \rightarrow, \leftrightarrow$ on the one hand, and $\sim$ on the other. The former are *binary* connectives: they have two slots that need to be filled, one on each side of the connective. '$P\&Q$' is a perfectly well-formed sentence. But '$P\&$' is not. Whereas 'It is raining and LA is cloudy' is a perfectly well-formed sentence, 'It is raining and' is not. The same goes for $\vee, \rightarrow, \leftrightarrow$: they all have two slots which need to be filled.

On the other hand, $\sim$ only has one slot to be filled; it is a *unary connective*. '$\sim P$' is a perfectly well-formed sentence. But '$P \sim Q$' is not. Whereas 'It's not raining' is a perfectly well-formed sentence, 'it is raining not Los Angeles is cloudy' is not.

One final point before moving on to truth-tables: we use parentheses to differentiate similar-looking sentences from each other. Here is an example: $\sim P\&Q$ and $\sim(P\&Q)$. Here is how they are different:

- $\sim P\&Q$ = It's not raining and LA is cloudy.
- $\sim(P\&Q)$ = It's not both the case that it is raining and LA is cloudy.

The first actually says that (1) it is not raining and (2) LA is cloudy. The second says that it's not both (1) raining and (2) cloudy in LA: maybe it's not raining, or it's not cloudy in LA, or neither, but it's definitely *not both* raining and cloudy in LA.

Hopefully you get the feel for how this little language works. If you want more detail, check out the 1^{st} appendix. But if you get the feel for how to build up sentences in PL, there's no need to read the 1^{st} appendix.

2 CONSTRUCTING TRUTH-TABLES

This section is broken down into three subsections. First, I'll give you the truth-tables for the connectives. Second, I'll show you how to construct truth-tables for more complex sentences. And third, I'll show you how to use truth-tables to check an argument for validity or invalidity.

2.1 Truth-Tables for the Connectives

Take 5-10 minutes, write these down on some note-cards, and memorize them:

Truth-table for &

P	Q	P	&	Q
T	T		T	
T	F		F	
F	T		F	
F	F		F	

Truth-table for $\vee$

P	Q	P	$\vee$	Q
T	T		T	
T	F		T	
F	T		T	
F	F		F	

Truth-table for $\rightarrow$

P	Q	P	$\rightarrow$	Q
T	T		T	
T	F		F	
F	T		T	
F	F		T	

Truth-table for $\leftrightarrow$

P	Q	P	$\leftrightarrow$	Q
T	T		T	
T	F		F	
F	T		F	
F	F		T	

Truth-table for $\sim$

P	$\sim$	P
T	F	
F	T	

These in effect tell you what the connectives mean. The connectives are really just functions: they take a sentence or two, consider whether they are true or false, and tell you whether the more complex sentence is true or false.

Let's start with an easy example: &. Suppose I gave you two true sentences, sentences like the following: 'LA is in California' and 'California is in the US'. These are both true. So ask yourself whether the following should be true or false: 'LA is in California and California is in the US'. Of course, this should be true. And the **Truth-table for** & tells us just that. It tells us that if P and Q are true, P&Q is true as well. But also notice that the

Truth-table for & tells us that any other combination for P and Q will result in P&Q being false.

Now think about the **Truth-table for** $\vee$. It says that the only way for a sentence of the form 'P $\vee$ Q' to be false is if both P and Q are themselves false. Consider these two sentences: 'Sarah Palin is Russian' and 'Sarah Palin is the Vice President'. Given that these are both false, ask yourself whether the following should be true or false: 'Either Sarah Palin is Russian or Sarah Palin is the Vice President'. Of course, this should be false. Neither P or Q are true, so P $\vee$ Q should be false as well. But also notice that the **Truth-table for** $\vee$ tells us that any other combination for P and Q will result in P $\vee$ Q being true. As long as one of P or Q is true, then P $\vee$ Q is true - and even if both P and Q are true, P $\vee$ Q is true.

Now think about the **Truth-table for** $\rightarrow$. This table says that the only way to make an 'if, then' sentence false is to make the first part true and the second part false. Here is an obvious example of a false 'if, then' sentence: If Obama is the President, then Obama is a trillionaire. While it is true that Obama is the president, it's false that he's a trillionaire. So the whole 'if, then' sentence is false as well.

Some people find the bottom two rows of the **Truth-table for** $\rightarrow$ to be a bit weird. Those rows say that if you consider an 'if, then' sentence where the first part of the sentence is false, then the whole thing is true. So take a false sentence, say, 'USC is located on Mars'. No matter what sentence comes after 'USC is located on Mars' in an 'if, then' sentence, the whole thing is true. So 'If USC is located on Mars, then sharks are cats' is true. And 'If USC is located on Mars, then California is in the US' is true as well. Stick anything on the right side of 'then' in the above sentence, and the whole thing is automatically true. Sometimes philosophers call these true 'if, then' sentences *vacuous truths*.

I'll give you one example for why you should think the **Truth-table for** $\rightarrow$ sort of makes sense. Suppose I tell you "If you get an A on every assignment this semester, then I will give you an A for the whole course." This is an 'if, then' sentence of the form 'P $\rightarrow$ Q' where P = 'you get an A on every assignment this semester', and Q = 'I will give you an A for the whole course'. Suppose you get an A on every assignment, but instead of giving you an A, I give you an F just to be funny. *I've lied to you*. So the whole sentence, 'If you get an A on every assignment this semester, then I will give you an A for the whole course.' is false. But suppose you don't try very hard this semester and you get a C on every assignment, thereby earning a C for the course. *Would I have lied to you*? Of course not. But would the following sentence be true or false: 'If you get an A on every assignment this semester, then I will give you an A for the whole course'? It is vacuously (automatically) True. You didn't get As for every assignment, so you didn't give me the chance to lie to you. So it's just vacuously true. Now: if this didn't help and just confused you, forget what I just said and simply memorize the **Truth-table for** $\rightarrow$.

Let's move on to the **Truth-table for** $\leftrightarrow$. This is a bit easier. An 'if and only if' sentence is true as long as both sides of the '$\leftrightarrow$' have the same truth-value, in other words, as long as they're *both* true or *both* false. But if one side has one truth-value and the other side doesn't, then the whole complex sentence is false.

Let's end with the easy truth-table: the **Truth-table for** $\sim$. This table is shorter because $\sim$ only applies to one sentence. As you might have guessed, $\sim$ takes a true sentence and makes it false, and take a false sentence and makes it true. Suppose P = 'USC is a university with cheap tuition rates'. P is clearly false. So what would $\sim$ P be? True. Because 'USC is not a university with cheap tuition rates' is true.

2.2 Constructing Truth-Tables for Complex Sentences

I now want to show you how to construct bigger truth-tables for more complex sentences of PL. Let's begin with an easy example: $P \vee \sim Q$.

Step 1: Write up the left-half of truth-table

P	Q	$P \vee \sim Q$
T	T	
T	F	
F	T	
F	F	

Step 2: Mark the main-connective with a $\downarrow$

P	Q	$P \overset{\downarrow}{\vee} \sim Q$
T	T	
T	F	
F	T	
F	F	

Finding the main-connective usually gets pretty easy. Parentheses often help. Here are some examples of sentences and their main-connectives.

- $P \overset{\downarrow}{\rightarrow} \sim Q$
- $\overset{\downarrow}{\sim} (P \rightarrow Q)$
- $(P \& Q) \overset{\downarrow}{\leftrightarrow} \sim (R \vee T)$
- $\overset{\downarrow}{\sim} ((P \& Q) \leftrightarrow \sim (R \vee T))$

If it's still not obvious how to find main-connectives, it's probably worth going through the 1st appendix.

Step 3: Copy the left columns over to the right columns

P	Q	P	$\overset{\downarrow}{\vee}$	$\sim$	Q
T	T	T			T
T	F	T			F
F	T	F			T
F	F	F			F

Step 4: Use the truth-table rules on the connectives, leaving the main-connective for last

Since $\vee$ is our main-connective, we'll work on the $\sim$ column first. Remember that the **Truth-table for** $\sim$ tells us to take the truth-value of the sentence on its right, and flip the truth value. So we look at the truth-values in the right-most column for Q, and write down the following:

P	Q	P	↓ ∨	~	Q
T	T	T		F	T
T	F	T		T	F
F	T	F		F	T
F	F	F		T	F

Step 5: Use the truth-table rules on the main connective

Since ∨ is our main-connective, we'll use the **Truth-table for the** ∨. This table takes two sentences (a sentence on the left and a sentence on the right) and spits out a truth-value. So for each (horizontal) row, we'll look at the truth-value for P (the one closest to ∨) and we'll look at the right to the ~ column (you can at this point forget about the Q column). So look at the first row. If P is true and ~ if F, then the whole thing is T. The final table should look like this:

P	Q	P	↓ ∨	~	Q
T	T	T	**T**	F	T
T	F	T	**T**	T	F
F	T	F	**F**	F	T
F	F	F	**T**	T	F

And we're done. The bold column is the truth-table for the PL sentence: P ∨ ~ Q.

Before moving on to the final subsection, let's do two more things. First, let's do one more (slightly more complicated) example. And second, let's talk about what to do when you have more than two simple PL sentence variables. Let's begin with the first. Let's write up the truth-table for: ~ ((P&Q) ↔ (Q ∨ P)).

Step 1: Write up the left-half of truth-table

P	Q	~	((P	&	Q)	↔	(Q	∨	P))
T	T								
T	F								
F	T								
F	F								

Step 2: Mark the main-connective

P	Q	↓ ~	((P	&	Q)	↔	(Q	∨	P))
T	T								
T	F								
F	T								
F	F								

Step 3: Copy the left columns over to the right columns

P	Q	$\overset{\downarrow}{\sim}$	((P	&	Q)	↔	(Q	∨	P))
T	T		T		T		T		T
T	F		T		F		F		T
F	T		F		T		T		F
F	F		F		F		F		F

Step 4: Use the truth-table rules on the connectives, leaving the main-connective for last

P	Q	$\overset{\downarrow}{\sim}$	((P	&	Q)	↔	(Q	∨	P))
T	T		T		T		T		T
T	F		T		F		F		T
F	T		F		T		T		F
F	F		F		F		F		F

This is slightly more complicated here than the last example, but not much. Forget about the ~ and look at the remaining sentence: ((P&Q) ↔ (Q ∨ P)). What would the main connective be for this sentence? It'd be the ↔. So if you'd like, you can mark it with another type of arrow (⇊):

P	Q	$\overset{\downarrow}{\sim}$	((P	&	Q)	$\overset{\downdownarrows}{\leftrightarrow}$	(Q	∨	P))
T	T		T		T		T		T
T	F		T		F		F		T
F	T		F		T		T		F
F	F		F		F		F		F

So work on the other connectives before working on the ↔, like this:

P	Q	$\overset{\downarrow}{\sim}$	((P	&	Q)	$\overset{\downdownarrows}{\leftrightarrow}$	(Q	∨	P))
T	T		T	T	T		T	T	T
T	F		T	F	F		F	T	T
F	T		F	F	T		T	T	F
F	F		F	F	F		F	F	F

Before working on the main connective, work on the ↔. Apply the **Truth-table rule for** ↔ by looking at each row for the &-column and the ∨-column, like this:

P	Q	$\overset{\downarrow}{\sim}$	((P	&	Q)	$\overset{\downdownarrows}{\leftrightarrow}$	(Q	∨	P))
T	T		T	T	T	T	T	T	T
T	F		T	F	F	F	F	T	T
F	T		F	F	T	F	T	T	F
F	F		F	F	F	T	F	F	F

Lastly, apply the **Truth-table rule for** ~ to the ↔-column, like this:

P	Q	↓ ~	((P	&	Q)	⇓ ↔	(Q	∨	P))
T	T	**F**	T	T	T	T	T	T	T
T	F	**T**	T	F	F	F	F	T	T
F	T	**T**	F	F	T	F	T	T	F
F	F	**F**	F	F	F	T	F	F	F

And we're done. The bold column is the truth-table for the PL sentence: $\sim((P\&Q) \leftrightarrow (Q \vee P))$.

Now, there's one more wrinkle to get a handle on. But it's easy. All of the sentences we've considered so far have two simple sentence variables: P and Q. And also notice that all of our complex truth-tables have had 4 rows. Things change when we consider sentences which contain more than 2 sentence variables, sentences like the following:

- $(P \vee Q) \vee R$
- $(P\&Q) \rightarrow (\sim R \vee \sim S)$
- $((P \leftrightarrow Q) \wedge (R \rightarrow S))\ \&\ \sim T$

The more sentence variables there are, the more rows there need to be: we'll need to consider more possible combinations of truth-values on the left hand side of our truth-tables. Here are the two rules for generating the right setup for a more complex truth-table:

Row-Rule 1: If your sentence has n sentence variables, write out 2^n rows.

Row-Rule 2: Start on the right-most row and alternate Ts and Fs by 1s. Then move over to the left one column, and alternate writing down Ts and Fs by 2. Then move over to the left and alternate Ts and Fs by 4, and so on (so next, you'd alternate by 8).

Consider the first example above: $(P \vee Q) \vee R$. Since there are 3 sentence variables, Row-Rule 1 tells us that there will be 8 rows. Now, according to Row-Rule 2, we write down the following:

P	Q	R	(P	∨	Q)	∨	R
		T					
		F					
		T					
		F					
		T					
		F					
		T					
		F					

Then, Row-Rule 2 tells us to write the following:

P	Q	R	(P	$\vee$	Q)	$\vee$	R
	T	T					
	T	F					
	F	T					
	F	F					
	T	T					
	T	F					
	F	T					
	F	F					

And lastly, Row-Rule 2 tells us to write the following:

P	Q	R	(P	$\vee$	Q)	$\vee$	R
T	T	T					
T	T	F					
T	F	T					
T	F	F					
F	T	T					
F	T	F					
F	F	T					
F	F	F					

And then we just follow our usual rules. Since we've written up the left-half of the truth table, our second rule says to mark the main-connective:

P	Q	R	(P	$\vee$	Q)	$\overset{\downarrow}{\vee}$	R
T	T	T					
T	T	F					
T	F	T					
T	F	F					
F	T	T					
F	T	F					
F	F	T					
F	F	F					

Third, copy the left columns over to the right columns:

P	Q	R	(P	$\vee$	Q)	$\overset{\downarrow}{\vee}$	R
T	T	T	T		T		T
T	T	F	T		T		F
T	F	T	T		F		T
T	F	F	T		F		F
F	T	T	F		T		T
F	T	F	F		T		F
F	F	T	F		F		T
F	F	F	F		F		F

Fourth, we work on the connectives, leaving the main connective for last, like this:

P	Q	R	(P	∨	Q)	↓ ∨	R
T	T	T	T	T	T		T
T	T	F	T	T	T		F
T	F	T	T	T	F		T
T	F	F	T	T	F		F
F	T	T	F	T	T		T
F	T	F	F	T	T		F
F	F	T	F	F	F		T
F	F	F	F	F	F		F

And lastly, we work on the main connective, by applying the **Truth-table rule for** ∨ to the left ∨-column and the R-column, like this:

P	Q	R	(P	∨	Q)	↓ ∨	R
T	T	T	T	T	T	**T**	T
T	T	F	T	T	T	**T**	F
T	F	T	T	T	F	**T**	T
T	F	F	T	T	F	**T**	F
F	T	T	F	T	T	**T**	T
F	T	F	F	T	T	**T**	F
F	F	T	F	F	F	**T**	T
F	F	F	F	F	F	**F**	F

And we're done. The bold column is the truth table for the PL sentence: $(P \vee Q) \vee R$.

2.3 Checking for Validity

We're almost done. We now need to put all of this together and show how to check whether an argument is valid or invalid. Suppose someone offers the following argument, and you want to check whether or not it is valid:

1. If the laws of physics determine everything, then we are not morally responsible for our actions.
2. It's not the case the we are not morally responsible for our actions.
3. Thus, the laws of physics do not determine everything.

This argument should be symbolized in PL as follows (using whatever letters you'd like; I'll use P and Q):

1. $P \to \sim Q$
2. $\sim\sim Q$
3. $\sim P$

P = 'the laws of physics determine everything' and Q = 'we are morally responsible for our actions'. Remember, sentence variables stand for simple English sentences which don't contain the connectives (and, or, if/then, if and only if, not). This is why we need to let Q represent 'we are morally responsible for our actions'. We represent the sentence 'we are not morally responsible for our actions' as $\sim Q$.

You might think this argument sounds valid. But is it? In what follows, I show you how to use the truth-tables to check.

Rule 1: Write out the truth-table for the entire argument like this:

P	Q	P	→	~	Q	~	~	Q	~	P

This might look complicated, but it's really not. What we have are just the usual left-most columns for our sentence variables, and then 3 truth-tables smashed together. The ones in the middle are the truth-tables for our two premises, and the one on the right is the truth-table for our conclusion. Moving on.

Rule 2: Use the 5 Steps for each individual truth-table.

Let's begin by filling in the left side of the truth table and marking the main connectives of the large truth-table, like this:

P	Q	P	↓ →	~	Q	↓ ~	~	Q	↓ ~	P
T	T									
T	F									
F	T									
F	F									

Then, let's copy and paste the left-most columns over into the large truth-table, like this:

P	Q	P	↓ →	~	Q	↓ ~	~	Q	↓ ~	P
T	T	T			T			T		T
T	F	T			F			F		T
F	T	F			T			T		F
F	F	F			T			F		F

Next, let's start by completing the truth-table for the first premise, like this:

P	Q	P	↓ →	~	Q	↓ ~	~	Q	↓ ~	P
T	T	T	**F**	F	T			T		T
T	F	T	**T**	T	F			F		T
F	T	F	**T**	F	T			T		F
F	F	F	**T**	F	T			F		F

And then, let's crank out the second premise, like this:

P	Q	P	$\overset{\downarrow}{\rightarrow}$	~	Q	$\overset{\downarrow}{\sim}$	~	Q	$\overset{\downarrow}{\sim}$	P
T	T	T	**F**	F	T	**T**	F	T		T
T	F	T	**T**	T	F	**F**	T	F		T
F	T	F	**T**	F	T	**T**	F	T		F
F	F	F	**T**	F	T	**F**	T	F		F

Lastly, let's crank out the conclusion, which will be extremely easy:

P	Q	P	$\overset{\downarrow}{\rightarrow}$	~	Q	$\overset{\downarrow}{\sim}$	~	Q	$\overset{\downarrow}{\sim}$	P
T	T	T	**F**	F	T	**T**	F	T	**F**	T
T	F	T	**T**	T	F	**F**	T	F	**F**	T
F	T	F	**T**	F	T	**T**	F	T	**T**	F
F	F	F	**T**	F	T	**F**	T	F	**T**	F

Now, we're ready for our final Rule.

Rule 3: Look at each (horizontal) row, and ask yourself the following question: on the row that I am looking at, are the truth-values of the premises true and the truth-value of the conclusion false? If so, the argument is **invalid**; if not, look at the next row. If you've gone through each row, and no row has Ts for all of the premises and an F for the conclusion, then the argument is **valid**.

So let's check our argument. On Row 1, the premises have an F and T, and the conclusion has an F. So move on. On Row 2, the premises have a T and F, and the conclusion an F. So move on. On Row 3, the premises have a T and T, but the conclusion has a T as well. So move on. On our last row, Row 4, the premises have a T and F, and the conclusion has a T. Since we didn't find one row on which the premises all had Ts while the conclusion had a F, the above argument is valid. So there is no way the premises can be true and the conclusion false.

Let's do one more example. Suppose someone gave you the following argument:

1. If Obama is a Muslim, he's not equipped to run our country.
2. Obama is not equipped to run our country.
3. Thus, Obama is a Muslim.

We should symbolize this in PL in the following way:

1. $P \rightarrow \sim Q$
2. $\sim Q$
3. P

Let's check whether or not this argument is valid, using the truth-table method. So according to Rule 1, write out the truth-table for the entire argument, like this:

P	Q	P	→	~	Q	~	Q	P

Rule 2 tells us to follow the steps for writing out the truth-tables for each of the premises and the conclusion. After we do that, we should have written down:

P	Q	P	$\overset{\downarrow}{\rightarrow}$	$\sim$	Q	$\overset{\downarrow}{\sim}$	Q	$\overset{\downarrow}{P}$
T	T	T	**F**	F	T	**F**	T	**T**
T	F	T	**T**	T	F	**T**	F	**T**
F	T	F	**T**	F	T	**F**	T	**F**
F	F	F	**T**	T	F	**T**	F	**F**

Now let's look at Rule 3. Are there are rows on which all of the premises have Ts while the conclusion an F? Yes. The last row, Row 4. Since there is at least one row on which the premises all have Ts and the conclusion F, the argument is invalid.

3 APPENDIX 1: PL SYNTAX

Coming soon.

4 APPENDIX 2: SUMMARY OF TRUTH-TABLE RULES

Coming soon.